Online Appendix for 'Forecast Uncertainty, Disagreement, and the Linear Pool'

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I Derivations for baseline example

Here we prove that D and S are independent for $\omega_1 = 0.5$. Consider the zero mean vector $W = \begin{bmatrix} X_1, & X_2, & U \end{bmatrix}'$ with variance-covariance-matrix

$$\Omega = \begin{bmatrix} \sigma_X^2 & \rho\sigma_X^2 & 0 \\ \rho\sigma_X^2 & \sigma_X^2 & 0 \\ 0 & 0 & \sigma_X^2 \end{bmatrix}.$$

We can write W as $W = C\ Z$, where C is the lower-diagonal Cholesky matrix of Ω, and Z is a trivariate vector of independent standard normals. Simple algebra yields that

$$C = \begin{bmatrix} \sigma_X & 0 & 0 \\ \rho\sigma_X & \sqrt{1-\rho^2}\sigma_X & 0 \\ 0 & 0 & \sigma_U \end{bmatrix}.$$

D is proportional to $(X_1 - X_2)^2$. We can write

$$(X_1 - X_2)^2 = (\underbrace{\begin{bmatrix} 1 & -1 & 0 \end{bmatrix}}_{=a'} \times W)^2 = Z'\ C'aa'C\ Z;$$

furthermore,

$$S = (\underbrace{\begin{bmatrix} \omega\ (1+\rho) & (1-\omega)\ (1+\rho) & 1 \end{bmatrix}}_{=b'} W)^2 = Z'\ C'bb'C\ Z.$$

A standard result (Craig, 1943) states that D and S are independent if $C'\ aa'\ CC'\ bb'\ C = 0$. Simple but tedious algebra shows that this condition is satisfied for $\omega_1 = 0.5$.

II (A)symmetry of the Centered Linear Pool

Here we collect a number of symmetry properties of the centered linear pool (CLP) density, f_{clp}. We first state these properties formally and then provide examples.

II.1 Formal properties

To introduce the relevant notation, let $f_i(x)$ be the ith density that enters the combination, and denote the mean of this density by m_i. The CLP density is given by

$$\begin{aligned} f_{clp}(x) &= \sum_{i=1}^{n} w_i \tilde{f}_i(x), \\ \tilde{f}_i(x) &= f_i\left(x - (m_{clp} - m_i)\right), \\ m_{clp} &= \sum_{i=1}^{n} w_i\ m_i. \end{aligned}$$

Note that $\tilde{f}_i(x)$ re-centers f_i at m_{clp} (in other words, it shifts the location of f_i such that its mean becomes equal to m_{clp}). We then have the following properties of f_{clp}.

A Suppose that all n component densities are symmetric around their respective median. Then the CLP density is symmetric around m_{clp}.

Proof: By symmetry, the mean m_i of the ith component density is also its median. By assumption, f_i is symmetric around m_i. The modified density $\tilde{f}_i(x)$ is symmetric around its mean (and median) m_{clp}. For $i = 1, \ldots, n$, we thus have that

$$\tilde{f}_i(x - m_{clp}) = \tilde{f}_i(-(x - m_{clp})).$$

Hence for the CLP it holds that

$$\sum_{i=1}^{n} w_i \tilde{f}_i(x - m_{clp}) = \sum_{i=1}^{n} w_i \tilde{f}_i(-(x - m_{clp})),$$

i.e., the CLP is symmetric around m_{clp}, as claimed.

B Suppose that all n component densities are unimodal and symmetric around their respective mode. Then the CLP density is unimodal and symmetric around its mode m_{clp}.

Proof: Since each component f_i is symmetric and unimodal, the mean, mode and median of f_i are all equal to m_i. The mean, mode and median of $\tilde{f}_i$ are all equal to m_{clp}. For any two points $x_1, x_2 \in \mathbb{R}$ with $m_{clp} < x_1 < x_2$ or $x_2 < x_1 < m_{clp}$, it thus holds that

$$\tilde{f}_i(x_1) \geq \tilde{f}_i(x_2),$$

and thus

$$f_{clp}(x_1) = \sum_{i=1}^{n} w_i \tilde{f}_i(x_1) \geq \sum_{i=1}^{n} w_i \tilde{f}_i(x_2) = f_{clp}(x_2),$$

i.e. the CLP has its unique mode at m_{clp}. The symmetry of f_{clp} around m_{clp} follows from part A.

II.2 Examples

We next provide examples for cases A, B and C. Table 1 describes the component densities that enter the combinaton. For simplicity, we use combination weights of 0.5 in each case. Furthermore, the CLP and LP densities have a mean of zero in each case. Figure 1 illustrates the CLP and LP densities, which are in line with the formal description of Section II.1:

- In case A, both components are themselves mixture densities. The first component density is symmetric around its median value of -1, whereas the second is symmetric around its median value of 1. The CLP density is bimodal and symmetric around zero, whereas the LP density is bimodal and asymmetric.

- In case B, the two components are Gaussian distributions with different means and variances. The CLP density is unimodal and symmetric around zero, whereas the LP density is unimodal and asymmetric.

- In case C, the components are asymmetric two-piece normal distributions. The densities of both the CLP and the LP are asymmetric.

			Symmetry?	
Case	First Component	Second Component	CLP	LP
A	$.5 \times \mathcal{N}(-3, 1) + .5 \times \mathcal{N}(1, 1)$	$.5 \times \mathcal{N}(-2, 4) + .5 \times \mathcal{N}(4, 4)$	Yes	No
B	$\mathcal{N}(-1, 1)$	$\mathcal{N}(1, 4)$	Yes	No
C	$2\text{p}\mathcal{N}(1.39, 5, 2)$	$2\text{p}\mathcal{N}(2.60, 3, 1)$	No	No

Table 1: Examples for cases A, B and C. All examples are parametrized such that the CLP and LP densities have mean zero. The notation $\mathcal{N}(a, b)$ indicates a normal distribution with mean a and variance b. The notation $2\text{p}\mathcal{N}(u, v, w)$ indicates a two-piece-normal distribution with parameters $\mu = u, \sigma_1 = v, \sigma_2 = w$ (see e.g. Wallis, 2014).

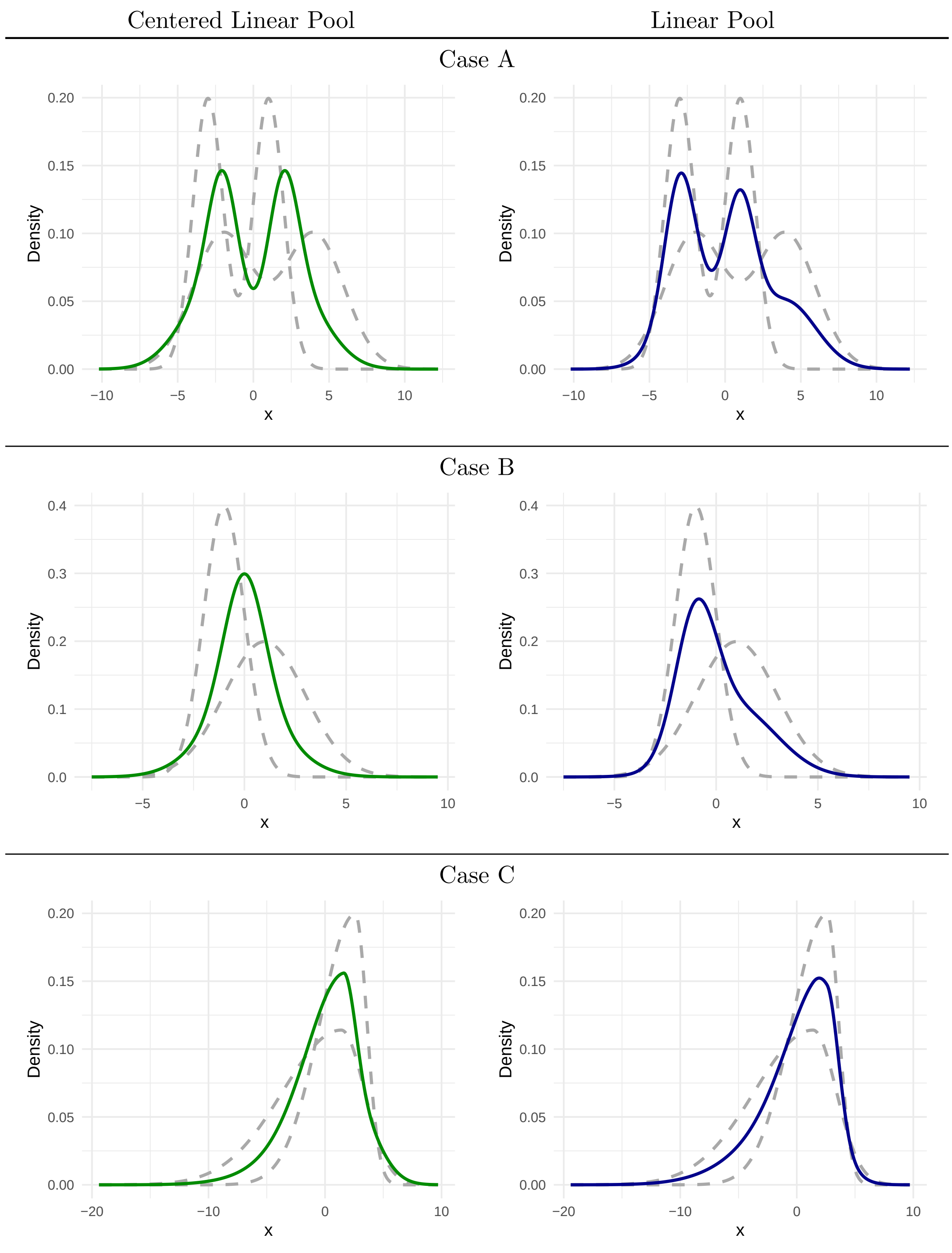

Figure 1: Densities of the centered linear pool and linear pool using based on equal weights for cases A, B and C.

III Results for intermediate horizons of Section 6

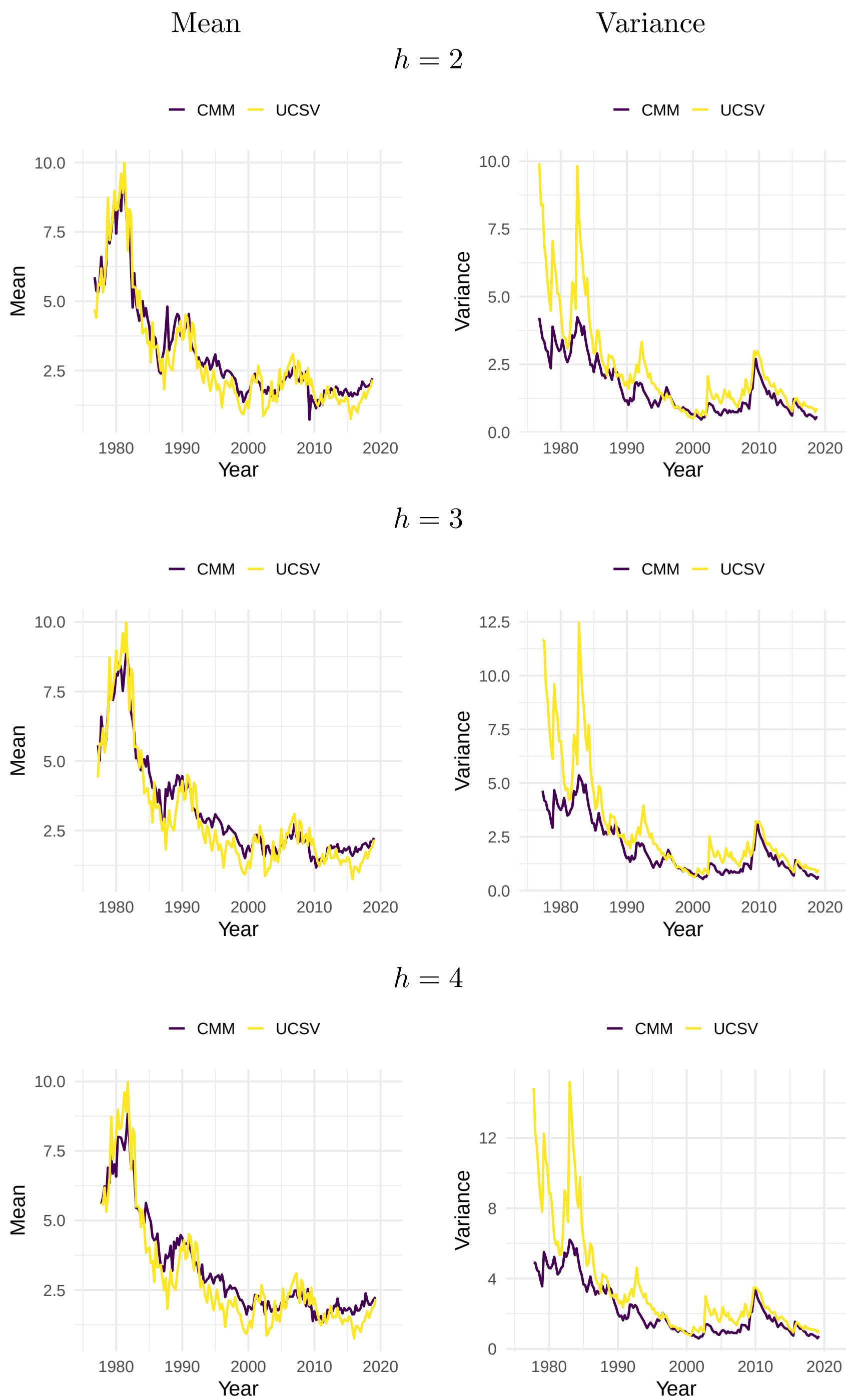

Figure 2: Mean (left column) and variance (right column) of the forecast distributions for the CMM and UCSV models. Rows correspond to different forecast horizons ($h = 2, 3, 4$). Evaluation sample ranges from 1976:Q2 to 2018:Q3.

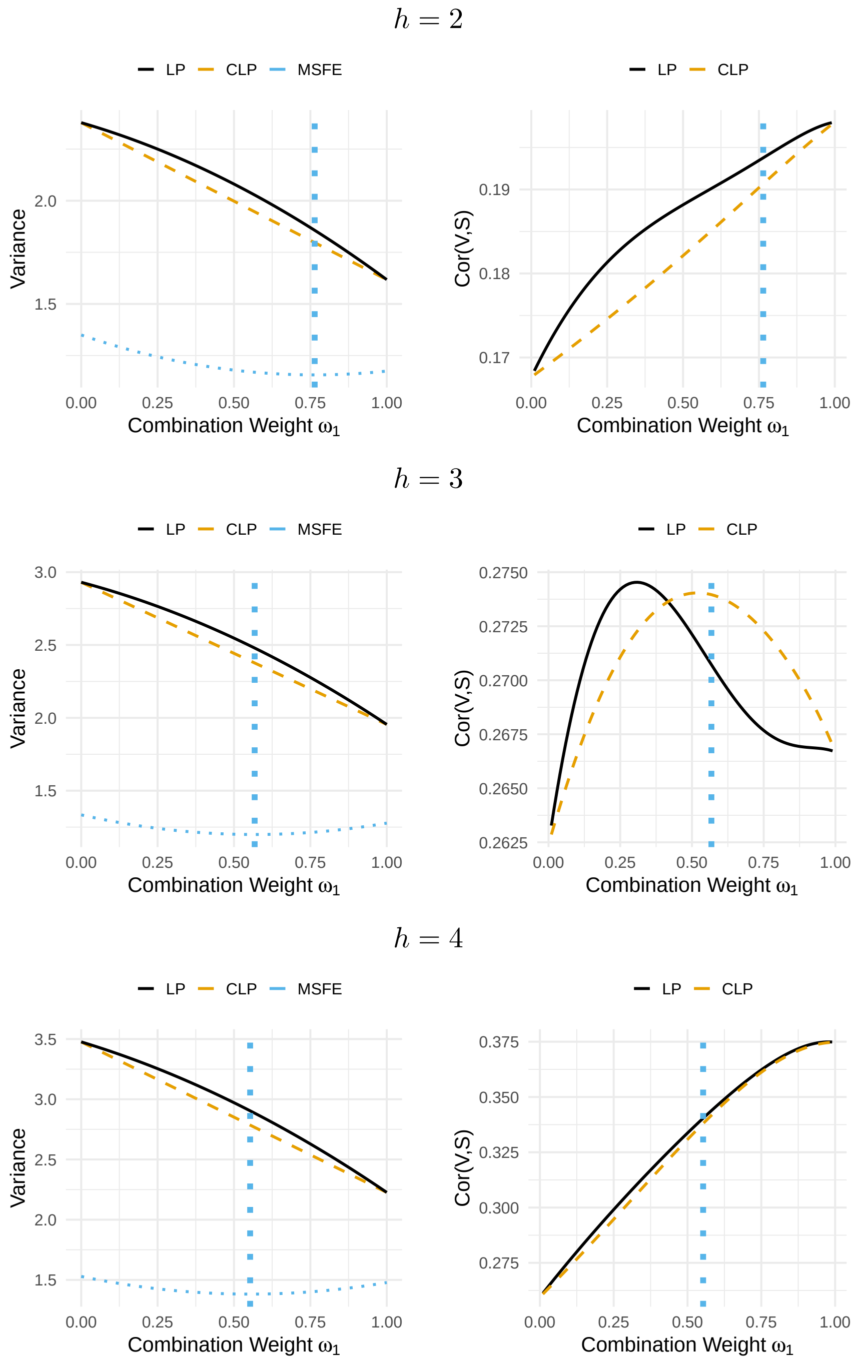

Figure 3: Variance forecasts and MSFE (left column), as well as correlation between variance forecasts (V_{lp} or V_{clp}) and squared forecast errors S (right column), plotted against the combination weight of the CMM model. MSFE-optimal weight is marked by blue vertical line in each plot. Rows correspond to forecast horizons ($h = 2, 3, 4$). Evaluation sample ranges from 1976:Q2 to 2018:Q3.

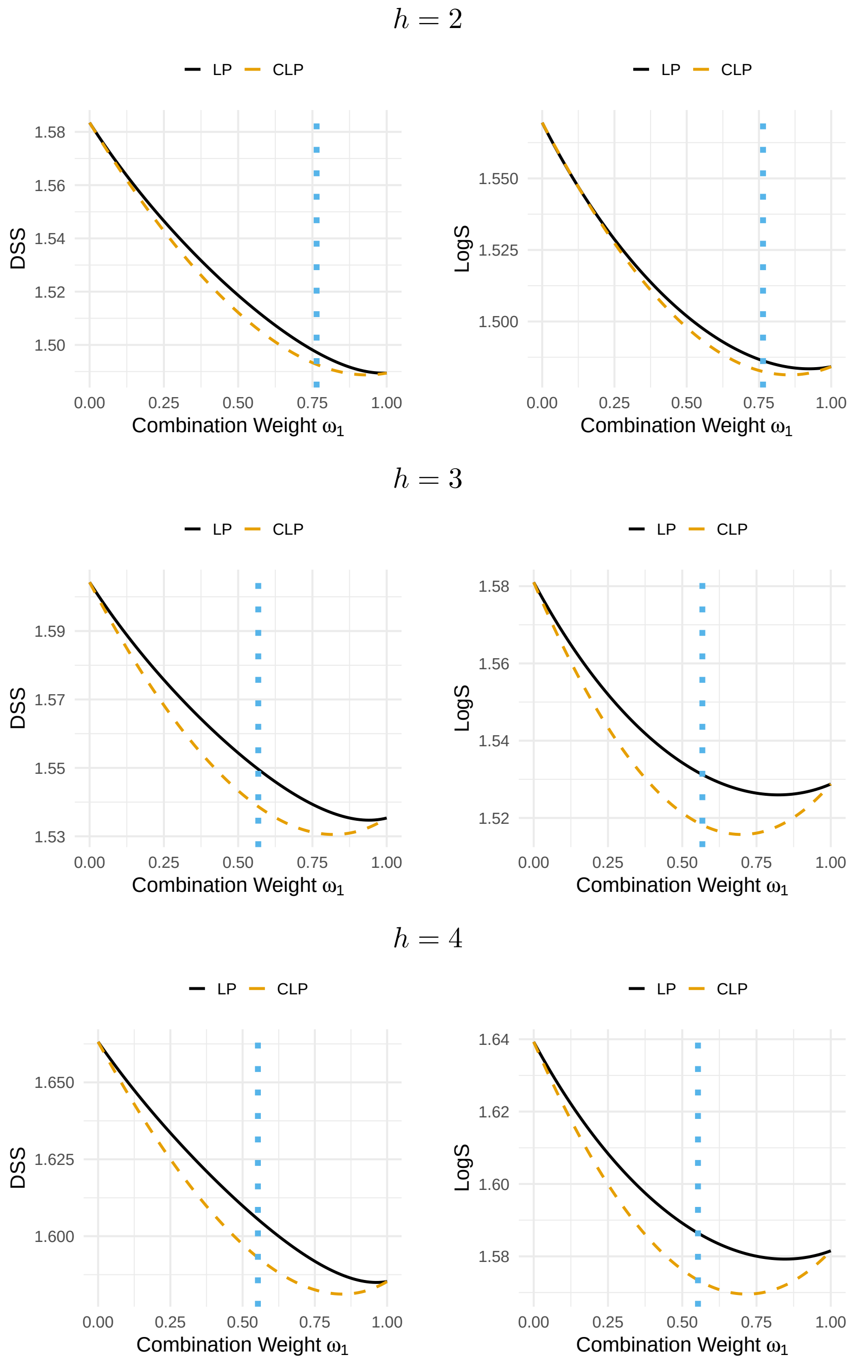

Figure 4: Dawid-Sebastiani score (left column) and logarithmic score (right column), plotted against the combination weight of the CMM model. Scores are in negative orientation, i.e. smaller scores are better. MSFE-optimal weight is marked by blue vertical line in each plot. Rows correspond to forecast horizons ($h = 2, 3, 4$). Evaluation sample ranges from 1976:Q2 to 2018:Q3.

References

Craig, A. T. (1943). Note on the independence of certain quadratic forms. *The Annals of Mathematical Statistics*, 14:195–197.

Wallis, K. F. (2014). The two-piece normal, binormal, or double Gaussian distribution: Its origin and rediscoveries. *Statistical Science*, 29:106–112.